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Abstract

If environmental quality positively affects the productivity of labor in R&D and pollution is caused by the use of a non-renewable resource, it is socially optimal to postpone extraction and to intertemporally adjust R&D effort.

Key words: Endogenous growth; Non-renewable resources; Environmental quality

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1 Introduction

We allow environmental quality to exert a positive effect on the productivity of labor in research and development (R&D) and study the implications of this assumption for the properties of the socially optimal dynamic path of the economy.

Our hypothesis is plausible since a clean and life-supporting environment is an essential factor for human activity in general. In this perspective the environment is an essential input for most creative economic activities and R&D in particular.

Our approach differs from common assumptions in the literature. Models of growth with environmental constraints emphasize the crucial role of R&D for allowing the economy to overcome the limits imposed by these constraints (Aghion and Howitt, 1998 ch.5). In the context of mounting pressure for environmental protection, R&D experiences a boom for two reasons. First, the value of innovations increases to the extent that these are relatively clean (a demand pull effect) (e.g. Hart 2004, Ricci 2007). Second, the (relative) production costs fall as factors of production exit relatively dirty sectors to the benefit of R&D (a favorable cost shift effect) (e.g. Elbasha and Roe, 1996).

In our opinion an additional aspect should be considered: Environmental degradation may increase R&D costs (an unfavorable cost shift). According to the hypothesis that we advance in this paper, a worsening state of ecosystems will call for a re-allocation of R&D effort.[2]

In a number of sectors, ecosystems provide valuable services not only to pro-

[2] Closest to our approach is the paper by van Ewijk and van Wijnbergen (1995). But whereas they consider human capital accumulation as the engine of growth and assume that pollution, as a flow, reduces the productivity of time devoted to education, we focus on non-rival knowledge as the growth engine and consider the damage from the stock of pollution.

duction processes but also at the stage of design and conception. In the pharmaceutical industry, for instance, biodiversity is a crucial asset, source of inspiration, and provider of test opportunities (Craft and Simpson, 2001). In general, an environment in a stable state provides potential access to a wealth of information and of possibilities to test theories and improve both fundamental and applied research. Environmental degradation may limit this function of ecosystems.

Our study assumes that the environment plays two distinct roles in the economic system. First it provides material inputs to production. Accordingly we assume that a non-renewable natural resource is a necessary input in manufacturing. Second, environmental quality is supposed to be a necessary input in R&D.

There is a trade-off between these two functions of the environment. The use of the natural resource implies polluting emissions that stock up and worsen the environmental quality. This impacts R&D negatively and thus potentially decreases economic growth.

Given that the polluting natural resource is non-renewable, its use must ultimately decline and the flow of polluting emissions shrink. Environmental quality will thus ultimately recover and approach some upper bound. Such an environmental Kuznets-curve suggests that there is scope for intertemporal substitution of the R&D effort, leading to richer dynamics than in related literature (e.g. Schou 2000).

To be able to study in isolation the role of environmental quality as a research asset, we abstract from any direct effect of environmental quality on social welfare, since in this case there is additional room for intertemporal substitution (e.g. Michel and Rotillon, 1995). For the same reason, we also abstract from any direct effect of environmental quality on total factor productivity in manufacturing.

With this framework, we obtain a five-dimensional dynamic system with R&D effort, depletion rate, resource stock, environmental quality, and the relative shadow price of environmental quality as endogenous variables. After presenting the model we derive the necessary conditions for optimality and study both local and global dynamics of the implied dynamic system.

We find that, as compared to the case where R&D is not directly affected by environmental quality, it is optimal to postpone extraction of the resource and that the optimal time path of R&D is non-monotonic. R&D starts above its asymptotic level but later undershoots it.

2 The model

Let L denote the constant size of population (and labor force). Consider the social planner's problem: choose $(L_{Yt}, R_t)_{t=0}^{\infty}$ so as to

$$\max U_0 = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} L e^{-\rho t} dt \quad \text{s.t.} \tag{1}$$

$$c_t = Y_t/L = A_t^\sigma L_{Yt}^\beta R_t^{1-\beta}/L, \quad 0 \le L_{Yt} \le L, \quad R_t \ge 0, \tag{2}$$

$$\dot{A}_t = \gamma A_t E_t^\varepsilon (L - L_{Yt}), \quad A_t \ge 0, \quad A_0 > 0, \text{ given}, \tag{3}$$

$$\dot{S}_t = -R_t, \quad S_t \ge 0, \quad S_0 > 0 \text{ given}, \tag{4}$$

$$\dot{E}_t = b\left(\bar{E} - E_t\right) - aR_t, \quad 0 < E_t \le \bar{E}, \quad E_0 \text{ given}, \tag{5}$$

where $L, \theta, \rho, \sigma, \gamma, \varepsilon, a, b, \bar{E} > 0$ and $\beta \in (0,1)$. The criterion function, (1), discounts future utility from per-capita consumption, c, by the rate of time preference, ρ. Production of a homogeneous manufacturing good, Y, employs two inputs: labor, L_Y, and a flow of an extracted resource, R, under constant returns to scale. Total factor productivity, A^σ, is increasing in the stock of technical knowledge, A, which grows through R&D according to (3).

The productivity of R&D is affected by two public goods: the stock of knowledge, proxied by cumulative R&D output, A, (see Romer, 1990); and the state of environmental quality, E.

The stock of the non-renewable resource is denoted by S and decreases over time, due to resource extraction, according to (4). Together with $S_t \geq 0$ this implies the restriction

$$\int_0^\infty R_t dt \leq S_0, . \tag{6}$$

Environmental quality evolves according to (5): it falls with extraction, R, and regenerates spontaneously at rate b. The maximum environmental quality is a given positive constant, $\bar{E}$. An ecological threshold, $E = 0$, exists which, if transgressed, implies disaster

3 Optimal dynamics

Until further notice, all variables (but not growth rates) are assumed positive. We suppress explicit dating of the variables. Let $g_x \equiv \dot{x}/x$ denote the growth rate of any variable x.

The current-value Hamiltonian for problem (1)-(5) is

$$H = \frac{c^{1-\theta} - 1}{1-\theta} L + \lambda_1 \gamma A E^{\varepsilon} (L - L_Y) - \lambda_2 R + \lambda_3 \left[b \left(\bar{E} - E \right) - aR \right],$$

where λ_1, λ_2, and λ_3 are the shadow prices of the state variables, A, S, and E, respectively. Necessary first-order conditions for an interior optimal solution are:

$$\frac{\partial H}{\partial L_Y} = c^{-\theta} \beta \frac{Y}{L_Y} - \lambda_1 \gamma A E^{\varepsilon} = 0, \tag{7}$$

$$\frac{\partial H}{\partial R} = c^{-\theta} (1 - \beta) \frac{Y}{R} - \lambda_2 - a\lambda_3 = 0, \tag{8}$$

$$\frac{\partial H}{\partial A} = c^{-\theta} \sigma \frac{Y}{A} + \lambda_1 \gamma E^{\varepsilon} (L - L_Y) = \rho \lambda_1 - \dot{\lambda}_1, \tag{9}$$

$$\frac{\partial H}{\partial S} = 0 = \rho \lambda_2 - \dot{\lambda}_2, \tag{10}$$

$$\frac{\partial H}{\partial E} = \lambda_1 \varepsilon \frac{\dot{A}}{E} - \lambda_3 b = \rho\lambda_3 - \dot{\lambda}_3. \tag{11}$$

Defining $h \equiv \lambda_3/\lambda_2$ (the shadow price of environmental quality in terms of the resource) and $u \equiv R/S$ (the depletion rate), we can derive the following dynamic system from the optimality conditions (7)-(11) and equations (2)-(5):[3]

$$\dot{S} = -uS, \tag{12}$$

$$\dot{h} = bh - \varepsilon \frac{\beta u S}{(1-\beta)L_Y E}(1+ah)(L-L_Y). \tag{13}$$

$$\dot{E} = b(\bar{E} - E) - auS. \tag{14}$$

$$\begin{aligned} \dot{u} = & \left\{ \theta u - (1-\theta)\,\beta\varepsilon b(\frac{\bar{E}}{E} - 1) + \left[\frac{1-\beta(1-\theta)}{1-\beta}\frac{L}{L_Y} - \frac{\theta}{1-\beta} \right] \beta\varepsilon a \frac{uS}{E} \right. \\ & \left. + \ (1-\theta)\,\sigma\gamma E^{\varepsilon} L - [1-\beta(1-\theta)]\, b \frac{ah}{1+ah} - \rho \right\} \frac{u}{\theta}, \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{L}_Y = & \left\{ \theta \frac{\sigma\gamma}{\beta} E^{\varepsilon} L_Y - [\beta + (1-\beta)\theta]\, \varepsilon b(\frac{\bar{E}}{E} - 1) + \left[(1-\theta)\,\beta \frac{L}{L_Y} + \theta \right] \varepsilon a \frac{uS}{E} \right. \\ & \left. + \ (1-\theta)\,\sigma\gamma E^{\varepsilon} L - (1-\theta)\,(1-\beta) b \frac{ah}{1+ah} - \rho \right\} \frac{L_Y}{\theta}. \end{aligned} \tag{16}$$

Equations (12)-(16) constitute a five-dimensional dynamic system in S, h, E, u, and L_Y. There are two pre-determined variables, S and E, and three jump variables, L_Y, u, and h.

A viable path (ensuring that $Y > 0$ for all t) is incompatible with a steady state. In fact constancy of E requires, by (5), $R = b(\bar{E} - E)/a$ constant, which contradicts (6) unless $R = 0$, thus $Y = 0$. We study instead a viable path that converges towards an *asymptotic steady state* (S^*, h^*, E^*, u^*, L_Y^*) for $t \to \infty$.

[3] Detailed derivation of the results of this article can be found in Groth and Ricci (2009).

If the following parametric restriction is satisfied

$$(1-\theta)\sigma\gamma\bar{E}^{\varepsilon}L < \rho < \sigma\gamma\bar{E}^{\varepsilon}L, \tag{A}$$

the system admits an asymptotic steady state where

$$S^* = h^* = 0, \quad E^* = \bar{E}, \quad u^* = \frac{1}{\theta}\left[(\theta-1)\sigma\gamma\bar{E}^{\varepsilon}L + \rho\right] > 0; \tag{17}$$

$$L_Y^* = \frac{\beta}{\theta\sigma\gamma\bar{E}^{\varepsilon}}\left[(\theta-1)\sigma\gamma\bar{E}^{\varepsilon}L + \rho\right] \in (0, L), \tag{18}$$

$$g_R^* = g_S^* = -u^* < 0, \tag{19}$$

$$g_A^* = \frac{1}{\theta}\left\{[\beta + (1-\beta)\theta]\,\gamma\bar{E}^{\varepsilon}L - \frac{\beta}{\sigma}\rho\right\} > 0, \tag{20}$$

$$g_c^* = g_Y^* = \frac{1}{\theta}(\sigma\gamma\bar{E}^{\varepsilon}L - \rho) > 0. \tag{21}$$

Linearizing the system we find that the Jacobian matrix evaluated at the asymptotic steady state has two negative and three positive eigenvalues. Hence, there exists a neighborhood of (S^*, E^*) such that when (S_0, E_0) belongs to this neighborhood, there is a unique path $(S_t, h_t, E_t, u_t, L_{Yt})$ converging towards the steady state.

To study the qualitative features of the global dynamics, we have run simulations for system (12)-(16) using the relaxation algorithm (Trimborn et al., 2008). Figures 1-2 show results from a simulation, based on the following parameter values: $\theta = 2.5$, $L = 1$, $\sigma = .75$, $\beta = .8$, $\gamma = 1.0$, $S_0 = 1.0$, $E_0 = \bar{E}$ $= 1.0$, $a = .05$, $b = .01$, and $\rho = .02$. The qualitative features of the results hold for alternative values of parameters. The case with a productive role of E in R&D ($\varepsilon = .25$) is compared with the case where labor productivity in R&D is independent of environmental quality ($\varepsilon = 0$).

As expected, resource depletion implies an environmental Kuznets curve, with an initial degradation of environmental quality followed by a recovery phase (left-hand panel of Figure 1). Similar dynamics for environmental quality hold in the case $\varepsilon = 0$. As indicated by the left-hand panels of Figure 2, this non-monotone evolution of E does not affect the optimal dynamics of control variables if $\varepsilon = 0$. When instead E is a productive asset in R&D, its non-monotone optimal path has implications for the optimal dynamics of the control variables u and L_Y as it can be seen from the right-hand panels of Figure 2.

First, notice from Figure 2 that with $\varepsilon > 0$, the resource depletion rate is persistently lower than in the case with $\varepsilon = 0$. This is due to extraction having a greater social cost when $\varepsilon > 0$. Not only does extraction now imply less resource availability in the future. It also lowers productivity of R&D.Interestingly, the optimal R&D effort evolves non-monotonically over time. As shown in the right-hand panel of Figure 1, R&D effort starts above its asymptotic level but then it undershoots it. This is the way the system strikes a balance between the incentive to take advantage of research opportunities when they are favorable and the desire for consumption smoothing. When environmental quality is worst, R&D has not yet reached its trough. This lag is due to the time-consuming nature of changes in the stock variable A (which governs total factor productivity in production).

4 Conclusion

Sustained growth is feasible and optimal even though the R&D sector rests on the natural capital. This is due to the fact that services from the environment to R&D are modeled as a renewable resource. The presence of this non-rival input to R&D affects the optimal policy. First, the rate of extraction of the polluting resource should be relatively low during the entire adjustment period. Second, R&D effort should evolve non-monotonically: given that resource exploitation implies first a deterioration and then a recovery of environmental quality, R&D effort adapts to changes in labor productivity in this sector.

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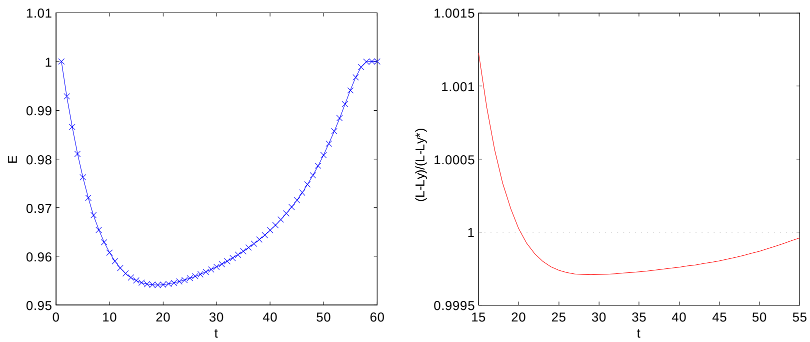

Fig. 1. Optimal time path of E (-x-) and R&D ($\frac{L-L_Y}{L-L_Y^*}$).

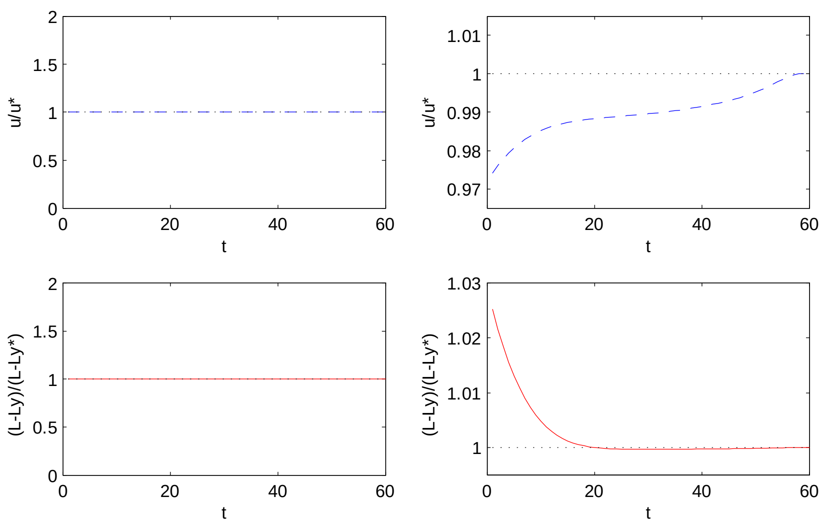

Fig. 2. Optimal time path of extraction (u/u^*) and R&D ($\frac{L-L_Y}{L-L_Y^*}$): case $\varepsilon = 0$ left-hand panels; case $\varepsilon = .25$ right-hand panels.

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Appendix not for publication

This appendix contains the detailed derivation of the results presented in the main text. This appendix will be available in the working paper version published on the internet sites of our departments. We first show how the dynamic system (12)-(16) is derived, next we consider the asymptotic steady state, and then the linearization of the system around the steady state in order to study the local dynamics. Finally we address the question how to establish that our candidate for an optimal solution, the unique converging path, is in fact optimal.

Dynamic system. Two growth accounting conditions obtained from the model are useful. First, (2) implies

$$g_c = g_Y = \sigma g_A + \beta g_{L_Y} + (1-\beta)\, g_R. \tag{22}$$

Second, (3) gives

$$g_A = \gamma E^{\varepsilon}(L - L_Y). \tag{23}$$

Ordering (7) and log-differentiating wrt. time, using $g_c = g_Y$, gives

$$(1-\theta) g_Y - g_{L_Y} = g_{\lambda_1} + \varepsilon g_E + g_A, \tag{24}$$

Ordering (9) yields

$$g_{\lambda_1} = \rho - c^{-\theta}\sigma \frac{Y}{\lambda_1 A} - \gamma E^{\varepsilon}(L - L_Y) = \rho - \frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} - g_A, \tag{25}$$

by (7) and (23). Now substitute (25) into (24) to get

$$g_{L_Y} = (1-\theta) g_Y - \rho + \frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} - \varepsilon g_E. \tag{26}$$

Combining (7) and (8) gives

$$\frac{(1-\beta)\, L_Y}{\beta R} = \frac{\lambda_2 + a\lambda_3}{\lambda_1 \gamma A E^{\varepsilon}} = \frac{1 + ah}{\frac{\lambda_1}{\lambda_2}\gamma A E^{\varepsilon}}. \tag{27}$$

Log-differentiating (27) wrt. time and ordering, using (9) and (10), leads to

$$g_R = g_{L_Y} - \frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} + \varepsilon g_E - \frac{a}{1+ah}\dot{h}. \tag{28}$$

Considering the stock value ratio $\lambda_1 A/(\lambda_3 E)$, we have

$$\frac{\lambda_1 A}{\lambda_3 E} \equiv \frac{\frac{\lambda_1}{\lambda_2} A}{hE} = \frac{\beta R(1+ah)}{(1-\beta) L_Y h E \gamma E^{\varepsilon}}, \tag{29}$$

in view of (27). Using $R \equiv uS$, (4), and (5) immediately yield (12) and (14), respectively.

By (10) and (11),

$$g_h = g_{\lambda_3} - g_{\lambda_2} = b - \varepsilon \frac{\lambda_1 A}{\lambda_3 E} g_A = b - \varepsilon \frac{\beta R(1+ah)}{(1-\beta)\, L_Y h E}(L - L_Y), \tag{30}$$

in view of (29) and (23). This explains (13). From (22) and (28),

$$g_Y = \sigma g_A + g_{L_Y} + (1-\beta)\left(-\frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} + \varepsilon g_E - \frac{a}{1+ah}\dot{h}\right). \tag{31}$$

Substituting this into (26) yields

$$\begin{aligned}
g_{L_Y} =& (1-\theta)\left[\sigma g_A + g_{L_Y} + (1-\beta)\left(-\frac{\sigma\gamma}{\beta} E^{\varepsilon} L_Y + \varepsilon g_E - \frac{a}{1+ah}\dot{h}\right)\right] \\
& -\rho + \frac{\sigma\gamma}{\beta} E^{\varepsilon} L_Y - \varepsilon g_E \\
=& (1-\theta)\left[\sigma\gamma E^{\varepsilon}(L - L_Y) + g_{L_Y} - \frac{\sigma}{\beta}\gamma E^{\varepsilon} L_Y + \sigma\gamma E^{\varepsilon} L_Y \right. \\
& \left. + (1-\beta)\left(\varepsilon g_E - \frac{a}{1+ah}\dot{h}\right)\right] - \rho + \frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} - \varepsilon g_E \qquad \text{(by (23))} \\
=& (1-\theta)\left[\sigma\gamma E^{\varepsilon} L + g_{L_Y} - (1-\beta)\frac{a}{1+ah}\dot{h}\right] - \rho + \frac{\theta\sigma\gamma}{\beta} E^{\varepsilon} L_Y \\
& + \left[(1-\theta)(1-\beta) - 1\right] \varepsilon g_E.
\end{aligned}$$

Solving for g_{L_Y} gives

$$\begin{aligned}
g_{L_Y} =& \frac{1}{\theta}\left\{(1-\theta)\left[\sigma\gamma E^{\varepsilon} L - (1-\beta)\frac{a}{1+ah}\dot{h}\right]\right. \\
& \left. -\rho + \frac{\theta\sigma\gamma}{\beta} E^{\varepsilon} L_Y - \left[\beta + \theta(1-\beta)\right]\varepsilon g_E\right\}.
\end{aligned} \tag{32}$$

Log-differentiating $u \equiv R/S$ wrt. t gives

$$\begin{aligned}
g_u =& g_R - g_S = g_R + u = g_{L_Y} - \frac{\sigma\gamma E^{\varepsilon} L_Y}{\beta} + \varepsilon g_E - \frac{a}{1+ah}\dot{h} + u \qquad \text{(from (28))} \\
=& \frac{\sigma\gamma}{\beta} E^{\varepsilon} L_Y - (\beta/\theta + 1 - \beta)\varepsilon g_E + \frac{1-\theta}{\theta}\sigma\gamma E^{\varepsilon} L - \frac{1-\theta}{\theta}(1-\beta)\frac{a}{1+ah}\dot{h} - \frac{\rho}{\theta} \\
& -\frac{\sigma\gamma}{\beta} E^{\varepsilon} L_Y + \varepsilon g_E - \frac{a}{1+ah}\dot{h} + u \qquad \text{(from (32))} \\
=& u - (\beta/\theta - \beta)\varepsilon g_E + \frac{1-\theta}{\theta}\sigma\gamma E^{\varepsilon} L - \left(\frac{1-\theta}{\theta}(1-\beta) + 1\right)\frac{a}{1+ah}\dot{h} - \frac{\rho}{\theta},
\end{aligned}$$

from which follows

$$\dot{u} = \left(u - \frac{1-\theta}{\theta}\beta\varepsilon g_E + \frac{1-\theta}{\theta}\sigma\gamma E^{\varepsilon} L - \frac{1-\beta(1-\theta)}{\theta}\frac{a}{1+ah}\dot{h} - \frac{\rho}{\theta} \right) u.$$

Taking into account (13) and (14) this can be written as (15). Finally, (32) can be written

$$\begin{aligned}\dot{L}_Y = \bigg[& \frac{\sigma\gamma}{\beta}E^{\varepsilon}L_Y - (\frac{\beta}{\theta} + 1 - \beta)\varepsilon g_E + \frac{1-\theta}{\theta}\sigma\gamma E^{\varepsilon} L \\ & - \frac{1-\theta}{\theta}(1-\beta)\frac{a}{1+ah}\dot{h} - \frac{\rho}{\theta} \bigg] L_Y.\end{aligned}$$

Taking into account (13) and (14) one obtains (16).

Asymptotic steady state. By the parameter restriction (A) follows $u^* > 0$, and so the asymptotic steady state has $S^* = 0$, in view of (12). Since $S^* = 0$, $\dot{h} = 0$ requires $h^* = 0$, in view of (13), and $\dot{E} = 0$ requires $E^* = \bar{E}$ according to (14). The remainder of (17) follows from (15). Further, by (16), L_Y^* must satisfy

$$\frac{\sigma\gamma}{\beta}\bar{E}^{\varepsilon}L_Y^* = \frac{1}{\theta}\left[(\theta - 1)\sigma\gamma\bar{E}^{\varepsilon}L + \rho \right] = u^*. \tag{33}$$

This can be rearranged, using (17), to obtain (18). Given that u^* is constant, (19) follows from (12). Then, by (22), (17), (18), and (19) we get

$$\begin{aligned} g_c^* = g_Y^* &= \sigma\gamma\bar{E}^{\varepsilon}(L - L_Y^*) + (1-\beta)\, g_R^* = \sigma\gamma\bar{E}^{\varepsilon}(L - L_Y^*) - (1-\beta)\, u^* \\ &= \sigma\gamma\bar{E}^{\varepsilon}L - \sigma\gamma\bar{E}^{\varepsilon}L_Y^* - (1-\beta)\, u^* = \sigma\gamma\bar{E}^{\varepsilon}L - u^* \\ &= \sigma\gamma\bar{E}^{\varepsilon}L - \frac{1}{\theta}\left[(\theta-1)\sigma\gamma\bar{E}^{\varepsilon}L + \rho \right], \end{aligned}$$

which can be reduced to (21). Finally, (20) is obtained using (18) in (23).

Linearization. The system can be approximated around the asymptotic steady state by a linearized system. The Jacobian matrix of the system (12)-(16), evaluated at the asymptotic steady state, is given by

$$\begin{array}{cccccc} & S & h & E & u & L_Y \\ \dot{S} & -u^* & 0 & 0 & 0 & 0 \\ \dot{h} & -(L - L_Y^*)\frac{\varepsilon\beta u^*}{(1-\beta)L_Y^*\bar{E}} & b & 0 & 0 & 0 \\ \dot{E} & -au^* & 0 & -b & 0 & 0 \\ \dot{u} & \{[1-\beta(1-\theta)]\, L - \theta L_Y^*\}\, \frac{\varepsilon\beta a u^{*2}}{(1-\beta)\theta\bar{E}L_Y^*} & -\left[1-\beta(1-\theta)\right]\frac{bau^*}{\theta} & j_{43} & u^* & 0 \\ \dot{L}_Y & \left[\beta\,(1-\theta)\, L + \theta L_Y^*\right] \frac{\varepsilon a u^*}{\bar{E}\theta} & -\frac{1-\theta}{\theta}(1-\beta)baL_Y^* & j_{53} & 0 & u^* \end{array}$$

where $j_{43} = \frac{1-\theta}{\theta}\left(\beta b + \sigma\gamma\bar{E}^{\varepsilon}L\right)\frac{\varepsilon u^*}{\bar{E}}$ and $j_{53} = \{[\beta + (1-\beta)\theta]\,\beta b + [\beta\,(1-\theta)\,L$ $+\,\theta L_Y^*]\,\sigma\gamma\bar{E}^{\varepsilon}\Big\}\,\frac{\varepsilon L_Y^*}{\beta\theta\bar{E}}$.

We see the Jacobian matrix is triangular so that the eigenvalues are the entries in the main diagonal. Two eigenvalues are negative and three are positive. This corresponds to the number of pre-determined variables (S and E) and jump variables (h, u, and L_Y), respectively.[4] Yet, since the linearized system is recursive, one should check whether also each of the subsystems in the causal ordering has a number of negative eigenvalues equal to the number of predetermined variables in that subsystem. Inspection of the Jacobian shows this to be the case. Thus, there exists a neighborhood of (S^*, E^*) such that when (S_0, E_0) belongs to this neighborhood, there is a unique path $(S_t, h_t, E_t, u_t, L_{Yt})$ converging towards the steady state.

Checking sufficient conditions. The transversality conditions of problem (1)-(5) are given by

$$\lim_{t\to\infty} \lambda_{1t}A_t e^{-\rho t} = 0, \tag{TVC1}$$
$$\lim_{t\to\infty} \lambda_{2t}S_t e^{-\rho t} = 0, \tag{TVC2}$$
$$\lim_{t\to\infty} \lambda_{3t}(\bar{E} - E_t)e^{-\rho t} \geq 0. \tag{TVC3}$$

Indeed, along the converging path, $\lambda_1 A e^{-\rho t}$ grows ultimately at the rate

$$g_{\lambda_1} + g_A^* - \rho = -\frac{\sigma\gamma}{\beta}\bar{E}^{\varepsilon}L_Y^* < 0,$$

by (25). Thus, the first transversality condition is satisfied. Along the converging path the second transversality condition also holds since $\lambda_2 S e^{-\rho t}$ grows ultimately at the rate

$$g_{\lambda_2} + g_S^* - \rho = -u^* < 0,$$

by (10) and (12). The third transversality condition is stated in a more general (and less common) form than the two others. This is because, seemingly, we cannot be sure that our candidate solution satisfies the more demanding condition $\lim_{t\to\infty} \lambda_{3t}E_t e^{-\rho t} = 0$. On the other hand, (TVC3) definitely holds, since $E_t \leq \bar{E}$ and $\lambda_{3t} > 0$ (and this is sufficient for our present purpose).

If only the maximized Hamiltonian were jointly concave in (A, E), our candidate solution would now satisfy a set of sufficient conditions for optimality according to Arrow's sufficiency theorem.[5] Unfortunately, however, the maximized Hamiltonian is not jointly concave in (A, E). Indeed, the maximized Hamiltonian is

[4] Interestingly, the eigenvalues appear in a symmetric way. In a pairwise manner they are of the same absolute size, but with opposite signs.

[5] See pp. 235-36 in Seierstad, A., and K. Sydsaeter (1987), *Optimal Control Theory with Economic Applications*, North-Holland: Amsterdam.

$$
\begin{aligned}
\hat{H}(A,S,E,\lambda_1,\lambda_2,\lambda_3,t) &= \max_{L_Y,R} H(A,S,E,L_Y,R,\lambda_1,\lambda_2,\lambda_3,t) \\
&= C_1 A^{-\frac{\beta(1-\theta)}{\theta}} E^{-\varepsilon\frac{\beta(1-\theta)}{\theta}} + \lambda_1 \gamma L A E^{\varepsilon} \\
&\quad - C_2 A^{-\left[\frac{\beta(1-\theta)}{\theta}+\sigma\right]} E^{-\varepsilon\frac{\beta(1-\theta)}{\theta}} - C_3,
\end{aligned}
$$

where C_1, C_2, and C_3 are positive coefficients not depending on A or E. We know the function $f(x,y) = x^{\alpha} y^{\beta}$ is concave if and only if

$$0 \leq \alpha \leq 1, \tag{34}$$
$$0 \leq \beta \leq 1, \quad \text{and} \tag{35}$$
$$\alpha + \beta \leq 1. \tag{36}$$

Thus, we come closest to concavity if $\theta = 1$. But even then, the term $\lambda_1 L A E^{\varepsilon}$ implies lack of joint concavity in (A, E). We therefore need to go via *existence* of an optimal solution.

Existence of an optimal solution. Given the parametric restriction (A), we can establish existence of an optimal solution by appealing to the existence theorem of d'Albis et al. (2008).[6] To apply this theorem, consider c and R as control variables and substitute $L_Y = A^{-\sigma/\beta} c^{1/\beta} R^{-(1-\beta)/\beta}$ into (3). Then the required joint concavity in the control variables of $u(\cdot)$ as well as the right-hand sides of (3), (4), and (5) is satisfied. And given (A), $\rho > (1-\theta) g_c^*$ holds and so the utility integral U_0 is bounded from above. As an implication, an optimal solution exists. Above we found that among the dynamic paths satisfying the necessary first-order conditions, there is only one converging path, all other paths being divergent. This leaves us with the converging path as the unique optimal solution.

[6] d'Albis, H., P. Gourdel and C. Le Van (2008), Existence of Solutions in Continuous-time Optimal Growth Models, *Economic Theory* 37: 321-333.